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Abstract

Quantum cognition is a recent and rapidely growing field. In this paper we develop an expected utility theory in a context of non-classical (quantum) uncertainty. We replace the classical state space with a Hilbert space which allows introducing the concept of quantum lottery. Within that framework we formulate sufficient and necessary axioms on preferences over quantum lotteries to establish a representation theorem. We show that demanding the consistency of choice behavior conditional on new information is equivalent to the von Neuman-Lüders postulate applied to beliefs. In our context, dynamic consistency is shown not to secure Savage's Sure Thing Principle (in its dynamic version). Finally, we discuss the interpretation and value of our results for rationality and behavioral economics.

1 Introduction

A central question facing any theory of decision making under uncertainty is how preferences are updated to incorporate new information. Dynamic consistency is the re-

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quirement that ex-ante contingent choices are respected by updated preferences. This consistency is implicit in the standard way of thinking about a dynamic choice problem as equivalent to a single ex ante choice to which one is committed, and is thus ubiquitous in economic modeling. Under (objective and subjective) expected utility, updating preferences by applying Bayes' rule is the standard way to update. Why is this so? Dynamic consistency is the primary justification for Bayesian updating as shown in Ghirardato (2002).

Dynamic consistency rationalizes the well-established theory of updating under expected utility, but what does it implies for updating in a more general uncertainty environment? This paper revisits Savage in a conditional world in a way similar to Ghirardato (2002) with the essential difference that we are dealing with non-classical (quantum) uncertainty. Fishburn (1970 p. 161) writes "It is generally assumed that (1) the decision maker does not know the "true state," (2) the act he selects has no effect on the state that obtains, and (3) the state that obtains affects the outcome of the decision in conjunction with the act selected". In this paper we are interested in situations where the choice of the act affects the state (we relax Fishburn's assumption (2)). A related line of motivation appeals to the growing interest for applications of elements of the mathematical formalism of Quantum Mechanics to psychology, social sciences and in particular in decision-making (see e.g., Brandenburger and La Mura (2015) and Busemeyer and Bruza (2012) for an overview of the field). The approach has shown successful in explaining a large variety of behavioral anomalies in decision-making ranging from cognitive dissonance, preference reversal, conjunction fallacy, disjunction effects to framing effects. This line of research is gaining recognition with a broader public as witnessed by recent publications in popular scientific magazines (e.g., Sciences et Vie, Sept. 2015).

Applying elements of the quantum formalism has also shown fruitful in explaining anomalies in information processing. There exists a large amount of evidence in psychology about order effects in the processing of information. In Section 7 we discuss

Bergus et al. (1998) experiment that exhibited a significant impact on physicians' diagnosis of the order in which they received information. Similar results exist regarding judicial decision-making. Various behavioral hypothesis have been proposed to explain order effect appealing to cognitive biases such as primacy, recency, saliency effects. Quantum cognition offers an explanation without appealing to any cognitive bias. Instead, it proposes that the mental image of the "world" relevant to decision-making behaves as a quantum-like object (see the Section 7 for a detailed presentation). The emphasis is on the "Bohr complementarity"[1] of a multiplicity of incompatible "perspectives"[2] on the world. Order effects arises as the expression of these incompatibilities and associated non-commutativity of measurements. In contrast, classical cognition assumes a unique perspective expressed by the existence of a single finest partition of the world which calls for Bayes' rule in updating.

The quantum cognitive explanation not only fits experimental data (Trueblood and Busemeyer (2012) and Wang et al. (2014)), it also lends itself to a meaningful psychological interpretation (Dubois and Lambert-Mogiliansky (2016)). But can people who hold a quantum like representation of the world behave dynamically consistently, can they be "rational"? Indeed the above mentioned behavioral explanations for order effects lead to dynamic inconsistency of choice behavior. In contrast with quantum cognition, order effect and dynamic consistency are compatible. We derive an updating rule that guarantees that choice behavior is consistent with a stable preference relationship when information processing exhibits order effects.

Besides the recent success in explaining behavioral phenomena, there are other reasons for turning to Quantum Mechanics in social sciences. Similarities between human sciences and quantum physics were early recognized by the founders of quantum mechan-

[1]A set of properties are said to be Bohr complementary when they inform about a system but yet cannot be measured simultaneously. Most well-known examples in Physics is position and momentum and the spin of a particle along different angles.

[2]The term "perspective" is used in the sense of Dubois and Lambert-Mogiliansky, it corresponds to a resolution of the Hilbert space that represents the agent's cognition. We provide a precise definition of "incomptibility" in the text.

ics, including Bohr and Heisenberg. In particular, Bohr was heavily influenced by the psychology and philosophy of knowledge of Harald Hoffding. A fundamental similarity stems from the fact that in both fields, the object of investigation cannot be separated from the process of investigation. This is expressed by the non-commutativity of measurements. The mathematical formalism of quantum Mechanics was developed to respond to that epistemological challenge: how can we "do science" about things that change as we try to learn about them (see Introduction in Bitbol (2009)). This historical and epistemological argument further justify exploring the properties of this mathematical formalism in the study of human behavioral phenomena.

In this article, we substitute the Boolean lattice of events with a more general lattice of projectors in the Hilbert space as the suitable framework for modelling decision-making. The notions are introduced progressively and require no previous knowledge of Quantum Mechanics or Hilbert spaces. We show that a natural definition of a quantum lottery allows for the formulation of decision theoretical axioms similar to the classical ones with one exception. We need axiom A5 that secures the stability of preferences over lotteries defined over different perspectives (resolutions of the state space). This axiom (that we labelled "no-framing") is trivially satisfied in the classical world (all lotteries can be expressed in a single finest partition(resolution) of the state space). We next show that the von Neumann-Lüders projection postulate of Quantum Mechanics used as an updating rule is both necessary and sufficient for dynamic consistency of preferences. In our context the von Neumann-Lüders postulate arises from purely behavioral considerations that is from a requirement of consistency applying to conditional (on new information) preference relations. Interestingly, the specificity of non-classical uncertainty (also referred to as "contextuality") is shown to imply a failure of the so-called "recursive dynamic consistency" (a dynamic version of the Sure Thing Principle). In a final section we discuss some implications of the results for rationality and behavioral economics.

There exists a few earlier works addressing quantum probabilities in the context of decision-making. These include Deutsch (1999), Pitowsky (2003), Lehrer and Shmaya (2006), Danilov and Lambert-Mogiliansky (2010) and Gyntelberg and Hansen (2012)). In particular Pitowky writes about "betting on quantum measurements" but he is not working with preference relations. Interestingly, he formulates a rule saying that the probability for any specific outcome is independent of the specific measurement that yields it as one of its possible results. This rule is very much in line with our axiom A5. Lehrer and Shamya propose a subjective approach to quantum probabilities but they do not work with quantum lotteries. Danilov and Lambert-Mogiliansky develop an expected utility theory in a general non-classical uncertainty context (ortho-modular lattices) but the analysis is static and in terms of utils. Gyltenberg and Hansen (2012) work with Hilbert space to develop an expected utility theory with subjective events. Their static setting shows similarities with ours. However their analysis appeals to a large number of axioms - 12 where we have 5 - and they do not address the issue of dynamic consistency.

The present work is a contribution to both decision theory and the foundations of quantum cognition. We extend previous works in two directions. First, we provide a complete characterization of expected utility theory under non-classical (quantum) uncertainty: we provide a concise and intuitive formulation of sufficient and necessary axioms in terms of preferences over quantum lotteries. Most importantly, this construction allows for a transparent characterization of dynamic consistency of choice behavior in such an environment. Finally, we discuss the value of the approach and illustrate it relying on experimental results from medical decision-making.

The paper proceeds as follows. First, we introduce the concept of quantum lottery which gives the opportunity to define basic elements of the mathematical formalism. Next, we provide an example of preferences over quantum lotteries and formulate the axioms needed to obtain a first theorem. In section 5 we proceed the other way around and characterize nice preferences to obtain our central representation theorem. In the next section we address the issue of information updating and formulate our theorem of dynamic consistency. Thereafter we discuss the value of our results and end with some concluding remarks.

2 Quantum lotteries

We are interested in a decision-maker's preferences over what we call quantum lotteries. In this section we define the notion of quantum or Q-lottery. But we shall start by reminding basic facts about roulette and so called "horse" lotteries. Any lottery entails an uncertain payoff: a prize is received depending on the realization of some event. We understand the term event as the outcome of a measurement. And the lotteries described below (roulette, horse and quantum lotteries) differ essentially in the type of measurement that is being performed.

Roulette lotteries. Assume that we have a set X of prizes. A roulette lottery (with prizes in X) is defined by a collection of prizes $x_1, ..., x_r$ together with the probabilities $p_1, ..., p_r$ (with $p_i \geq 0$ and $\sum_i p_i = 1$) for obtaining the corresponding prize. We could think of it in the following way: a measurement in the form of a 'roulette' is performed and gives an outcome in the set $\{1, ..., r\}$. The probability of outcome i is p_i and, depending on the outcome of this 'measurement', a prize x_i is paid.

Obviously, such lotteries can be identified with (simple) probabilistic measures on the set X. We denote by $\Delta(X)$ the set of such measures (or lotteries). Under well-known conditions, von Neumann and Morgenstein obtained that the utility of a lottery $l = (x_1, p_1; ...; x_r, p_r)$ for a decision-maker (DM) is defined by a number $U(l) = \sum_i p_i u(x_i)$ where $u : X \to \mathbb{R}$ is a 'utility function' defined on the set X of all possible prizes.

Horse lotteries.

The next concept is that of a 'horse lottery' or an 'act' in Savage's terminology. The decision-maker assigns probabilities $p_1, ..., p_r$ (with $p_i \geq 0$ and $\sum_i p_i = 1$) to the different outcomes. A horse lottery is a function $f : S \to X$ from the set S of states of nature to the set X of prizes. A measurement is performed in the form of a 'horse race' and, depending on the result of this measurement, the corresponding prize is paid.

Again under suitable conditions the utility of a horse lottery f can be written as $U(f) = \sum_s p_s u(f(s))$, where $u : X \to \mathbb{R}$ is a utility function, and p is a (subjective) probability measure on the set S. A considerable simplification was achieved by Anscombe and Aumann when taking roulette lotteries as prizes. They define a horse lottery as a function $f : S \to \Delta(X)$. In particular, this allows to view S as a set with a finite number of elements.

In order to smoothly move over to quantum lotteries, it is convenient to present horse lotteries slightly differently. We interpret any subset of S as an *event* and a *partition* is a (finite) collection of events $A_1, ..., A_r$ (r is an arbitrary number) which are mutually exclusive and exhaustive. That is $A_i \cap A_j = \emptyset$ if $i \neq j$, and $A_1 \cup ... \cup A_r = S$. In these terms a horse lottery is described by a collection $x_1, ..., x_r$ of prizes; prize x_i is given if the event A_i occurs.

Quantum lotteries.

A quantum lottery is also a bet on the outcome of a measurement, but now a quantum one. A measurement of some 'observable' is performed, and, depending on the result obtained, our DM receives some prize. To formalize the notion of quantum measurement we have to modify the previously defined notions of state, event and partition. We do this in the following way. The set S is replaced by some Hilbert space H. An event is a linear subspace in H. Finally, the notion of partition is replaced by the notion of orthogonal decomposition, that is a resolution of H as a (finite) sum of orthogonal events-subspaces, $H = V_1 \oplus ... \oplus V_r$ (r again is an arbitrary number). We should understand this orthogonal decomposition as a measurement; if such a measurement is performed then one and only one of the events $A_1, ..., A_r$ occurs. If the prize x_i is paid when event A_i occurs, we speak about a quantum lottery.

The main difference with the standard (classical) state space model is that the Hilbert space model allows for measurements that cannot be performed simultaneously; they are incompatible in general. And, as a consequence of the performance of a measurement, the state of the system can change. Below we give precise definitions, after recalling some elementary notions about Hilbert spaces.

Hilbert spaces.

Consider the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. In the latter case $\bar{z}$ denotes the complex conjugate of (complex) number z. The notions introduced below can be defined for complex and real numbers in a similar manner. We limit ourselves to the complex case, the real case is even simpler.

Definition. Let H be a vector space over $\mathbb{C}$. An *Hermitian form* on H is a mapping $(.,.) : H \times H \to \mathbb{C}$ such that a) it is linear in the first argument; b) $(v, w) = \overline{(w, v)}$. In particular, (v, v) is a real number. c) $(v, v) \geq 0$ for any $v \in H$, and $= 0$ only for $v = 0$.

Vectors v and w are called *orthogonal* if $(v, w) = 0$; in this case $(w, v) = 0$ as well.

A *Hilbert space* is a vector space H endowed with a Hermitian form, which is complete relatively to the norm $|v| = \sqrt{(v, v)}$. In the following we shall only be dealing with finite dimensional space so the condition of completeness is automatically fulfilled. Further we let H denote some Hilbert space.

Definition. An *event* is a closed vector subspace of H.[3]

The set of events is denoted by $\mathbb{P}(\mathcal{H})$. The operations of intersection and sum give a lattice structure in $\mathbb{P}(\mathcal{H})$. For an event V, the *opposite* event is given by subspace $V^{\perp}$ consisting of vectors $w \in H$ orthogonal to all vectors from V. $V = (0)$ is the least ('impossible') event and $V = H$ is the most ('trivial') event. Two events V and W are *orthogonal* if every vector from V is orthogonal to every vector from W.

Definition. An *orthogonal decomposition* (OD) of H is a finite family $(V_i, i \in I)$ of events V_i, such that

[3]In the finite-dimensional case all subspaces are closed.

a) V_i and V_j are orthogonal for $i \neq j$;

b) the sum of all V_i is equal to H.

Example 1. Let V be an event. Then the pair V and $V^{\perp}$ forms a ('dichotomous') orthogonal decomposition of H.

Given an orthogonal decomposition $(V_i, i \in I)$, we can consider V_i as an exhaustive collection of mutually exclusive events. Therefore we can form a "quantum" lottery which gives some prize x_i when event V_i occurs: $V_i \Rightarrow x_i, i \in I$. The system of events (V_i) is viewed as a measurement device. The performance of a measurement with this device gives one and only one of the event V_i; in this case our decision-maker gets prize x_i.

Definition. A *quantum lottery* (Q-lottery) is an OD $(V_i, i \in I)$ with an accompanying collection of prizes $(x_i, i \in I)$. We denote such a Q-lottery as $\sum_i x_i \otimes V_i$. The OD $(V_i, i \in I)$ is called the *base* of the lottery.

As Anscombe-Aumann, we shall allow roulette lotteries as prizes. The set of Q-lotteries (with prizes from $\Delta(X)$) will be denoted as $\mathbf{QL}(H, \Delta(X))$ or simply as $\mathbf{QL}(H)$ because a specification of the set X does not play an essential role. We would like to explore 'nice' preference relations on the set $\mathbf{QL}(H)$. In the next section we provide a way to construct such preferences and investigate their properties. Thereafter we proceed the other way around and characterize the set of nice preferences.

3 Construction of "nice" preferences

Let us recall that the 'utility' of a horse lottery is constructed by means of two ingredients: a linear (affine) utility function $u : \Delta(X) \rightarrow \mathbb{R}$ and a probability p on the Boolean lattice of subsets of S. To realize this program for Q-lotteries, we have to define the notion of 'probability' on the lattice $\mathbb{P}(H)$.

Definition. A *probability measure* on the lattice $\mathbb{P}(H)$ of events in a Hilbert space H is a function $\mu : \mathbb{P}(H) \rightarrow \mathbb{R}$ such that

a) $\mu(V) \geq 0$ for any event V;

b) $\sum_i \mu(V_i) = 1$ for any OD $(V_i, i \in I)$.

Note, that then $\mu(H) = 1$ and $\mu(V) + \mu(W) = \mu(V \oplus W)$ for orthogonal V and W.

We next provide a couple of examples of probability measures on $\mathbb{P}(H)$. Here and further we assume that H is a finite-dimensional Hilbert space. For an event V we denote by $dim(V)$ the dimension of V. We give two examples of probability measures.

Example 2. The first example is very simple. Set $\mu(V) = dim(V)/dim\,(H)$. It is easy to check that μ is a probability measure, a uniform distribution.

Example 3. More interesting is our second example. Let e be a vector in H with unit length, $(e, e) = 1$. For an event V, let e_V and $e_V^{\perp}$ be the unique vectors in V and $V^{\perp}$ respectively such that $e = e_V + e_V^{\perp}$. Then define $\mu_e(V)$ as (e_V, e). We assert that μ_e is a probability measure. Indeed,

$$(e_V, e) = (e_V, e_V + e_V^{\perp}) = (e_V, e_V) + (e_V, e_V^{\perp}) = (e_V, e_V) \geq 0.$$

In other words, $\mu_e(V)$ is the square of the length of e_V. Moreover, if $(V_i, i \in I))$ is an OD, then

$$\sum_i \mu_e(V_i) = \sum_i (e_{V_i}, e) = (\sum_i e_{V_i}, e) = (e, e) = 1,$$

This equality can be considered as Pythagore's theorem. This also referred to as Born rule in Quantum Mechanics.

Given a linear 'utility function' u on $\Delta\,(X)$ and a probability measure μ on $\mathbb{P}(H)$, we can define the '(u, μ)-utility', $U_{u,\mu}(\sigma)$ of any Q-lottery $\sigma = \sum_i l_i \otimes V_i$ as

$$U_{u,\mu}(\sigma) = \sum_i u(l_i)\mu(V_i),$$

and the corresponding preference relation $\preceq = \preceq_{u,\mu}$ on $\mathbf{QL}(H)$. Namely, for Q-lotteries we have $\sigma \preceq \tau$ if $U_{(u,\mu)}(\sigma) \leq U_{(u,\mu)}(\tau)$. Below we list some 'nice' properties that a preference relation $\preceq$ of the form $\preceq_{(u,\mu)}$ satisfies.

Weak order.

A1. *The preference relation $\preceq$ is a weak order, that is complete and transitive.*

This follows from its representation via the utility $U_{(u,\mu)}$.

To formulate the next three properties we need the notion of a mixture (or a convex combination) of Q-lotteries having the same base $(V_i, i \in I)$. Let $\sigma = \sum_i l_i \otimes V_i$ and $\tau = \sum_i m_i \otimes V_i$ be two Q-lotteries, and $\alpha \in [0,1]$; we define their mixture $\alpha\sigma + (1-\alpha)\tau$ as Q-lottery $\sum_i (\alpha l_i + (1-\alpha)m_i) \otimes V_i$.

Independence.

A2. *Let σ, τ, φ be Q-lotteries with the same base, and $\alpha \in [0,1]$. If $\sigma \preceq \tau$ then $\alpha\sigma + (1-\alpha)\varphi \preceq \alpha\tau + (1-\alpha)\varphi$.*

It follows from the equality $U(\alpha\sigma + (1-\alpha)\varphi) = \alpha U(\sigma) + (1-\alpha)U(\varphi)$ which is a consequence of the linearity of the utility U

Continuity.

A3. *Let σ, τ, φ be Q-lotteries with σ, τ defined on the same base, and $\sigma \prec \varphi \prec \tau$. Then there exists α and β $(0 < \alpha, \beta < 1)$ such that $\alpha\sigma + (1-\alpha)\tau \prec \varphi$ and $\varphi \prec \beta\sigma + (1-\beta)\tau$.*

Indeed, $U(\sigma) < U(\varphi) < U(\tau)$. Therefore $U(\varphi) > \alpha U(\sigma) + (1-\alpha)U(\tau)$ for some $\alpha \in]0,1[$ and $U(\varphi) < \beta U(\sigma) + (1-\beta)U(\tau)$ for some $\beta \in]0,1[$. So

$U(\varphi) > U(\alpha\sigma + (1-\alpha)\tau)$ and $U(\varphi) < U(\beta\sigma + (1-\beta)\tau)$. Note, that here we only use the fact that σ and τ have the same base.

Monotonicity.

To formulate this property, we define (with help of the preference relation $\preceq$ on $\mathbf{QL}(H)$) the *derived* preference relation $\preceq_L$ on the set of ordinary lotteries $\Delta(X)$. For lotteries l and m we set $l \preceq_L m$ if $l \otimes H \preceq m \otimes H$. Here $l \otimes H$ denotes a trivial Q-lottery, getting with certainty the prize l, and similarly for $m \otimes H$.

Note that the preference $\preceq_L$ on $\Delta(X)$ is represented by the function u. Indeed, $U_{u,\mu}(l \otimes H) = u(l)\mu(H) = u(l)$.

A4. *Let* $\sigma = \sum_i l_i \otimes V_i$ *and* $\tau = \sum_i m_i \otimes V_i$ *be Q-lotteries with the same base. If* $l_i \preceq_L m_i$ *for any* $i \in I$ *then* $\sigma \preceq \tau$.

This property can be considered as a version of the sure-thing principle of Savage. It is a trivial consequence of the non-negativity of $\mu(V_i)$.

No-framing.

The previous three properties were about Q-lotteries defined on *the same base or OD*. The last property **A5** binds lotteries with different (although "compatible") ODs. Let $\sigma = \sum_i l_i \otimes V_i$ and $\tau = \sum_j m_j \otimes W_j$ be two Q-lotteries based on ODs $(V_i, i \in I)$ and $(V_j, j \in J)$ respectively.

Definition. We say that τ is a *refinement* of σ, if there exists a mapping $\psi : J \to I$ such that (for any $j \in J$)

a) $W_j \subseteq V_{\psi(j)}$;

b) $m_j = l_{\psi(j)}$.

In words, the Q-lottery τ refines (subdivides) every event V_i but leaves the associated prizes unchanged. Indeed, we claim that

Lemma 1. *For any* $i \in I$, $V_i = \bigoplus_{j,\psi(j)=i} W_j$.

Proof. Fix $i \in I$ and let $J_i = \psi^{-1}(\{i\})$. We show that $V_i = \bigoplus_{j \in J_i} W_j$. Since $W_j \subseteq V_i$ for any $j \in J_i$, we clearly have $V_i \supseteq \bigoplus_{j \in J_i} W_j$. Moreover, let $v \in V_i$, and write in the form $v = \sum_{j \in J} w_j$ with each $w_j \in W_j$. To conclude, it is sufficient to show that $w_j = 0$ for any $j \notin J_i$. Fix $j \notin J_i$ and $i' = \psi(j) \neq i$. Then, $v \in V_i$ and $w_j \in W_J \subseteq V_{i'}$ so that v and w_j are orthogonal. But then $(w_j, w_j) = (v, w_j) = 0$ so that $w_j = 0$. So indeed $v \in \bigoplus_{j \in J_i} W_j$. ■

A5. *If* τ *is a refinement of* σ *then these Q-lotteries are equivalent for the DM:* $\sigma \sim \tau$.

Indeed, as we have seen, $V_i = \bigoplus_{j,i=\psi(j)} W_j$. Therefore,

$$U(\sigma) = \sum_i u(l_i)\mu(V_i) = \sum_i u(l_i)\mu(\bigoplus_{j,i=\psi(j)} W_j) = \sum_i u(l_i) \sum_{j,i=\psi(j)} \mu(W_j)$$

$$= \sum_i \sum_{j,i=\psi(j)} u(l_i)\mu(W_j) = \sum_i \sum_{j,i=\psi(j)} u(m_j)\mu(W_j) = \sum_j u(m_j)\mu(W_j) = U(\tau).$$

Interestingly, this axiom is implicit in the Savage and Anscombe-Aumann frameworks. However in generalizations of these frameworks it must be imposed explicitly see e.g., Cohen and Jaffray (1980). They formulate an axiom of "non influence of formalization" very similar to our axiom A5. There are also other works that reject that axiom in order to allow for framing effects see Ahn and Ergin (2010).

Definition. A preference relation $\preceq$ on the set $\mathbf{QL}(H)$ of Q-lotteries is *nice* if it is endowed with properties $\mathbf{A1} - \mathbf{A5}$ (or satisfies the axioms $\mathbf{A1} - \mathbf{A5}$).

The arguments above imply

Proposition 1. *A preference relation $\preceq_{u,\mu}$ on $\mathbf{QL}(H)$ constructed by means of an affine utility function $u : \Delta(X) \to \mathbb{R}$ and a probabilistic measure μ on $\mathbb{P}(H)$ is nice.*

In the next section we show that the inverse also is true.

4 Characterization of nice preferences on Q-lotteries

The previous section identified a number of axioms on preferences necessary for an expected utility representation (Proposition 1). The Theorem below shows that these axioms are also sufficient. Thus the axioms characterizing expected utility over quantum lotteries are very similar to the ones used by Anscombe-Aumann in a classical uncertainty context.

Theorem 1. *Suppose that $\preceq$ is a nice preference relation on the set $\mathbf{QL}(H)$. Then there exists an affine function u on $\Delta(X)$ and a probabilistic measure μ on $\mathbb{P}(H)$ such that $\preceq = \preceq_{u,\mu}$.*

Moreover, if the preference $\preceq$ is not trivial (that is there exist Q-lotteries σ and τ such that $\sigma \prec \tau$) then μ is unique and u is unique up to positive affine transformation.

Proof of Theorem 1. If preference $\preceq$ is trivial then the assertion is trivially true. Indeed, we can take u to be constant and take an arbitrary μ. So, from now on, we assume that preference $\preceq$ is nontrivial. Let us remind of the derived relation $\preceq_L$ on $\Delta(X)$: $l \preceq m$ iff $l \otimes H \preceq m \otimes H$. As a consequence of nontriviality, **A4** and **A5**, we obtain that the preference $\preceq_L$ is not trivial; we fix two (ordinary) lotteries l_* and l^* such that $l_* \prec_L l^*$. A function $u : \Delta(X) \to \mathbf{R}$ is said to be *normalized* if $u(l_*) = 0$ and $u(l^*) = 1$.

Claim 1. *For any OD $\mathcal{V} = (V_1, \ldots, V_n)$, there exists a normalized function $u_{\mathcal{V}}$ on $\Delta(X)$ and a probability measure $\mu_{\mathcal{V}}$ on the set of indexes I such that, for any $l_1, \ldots l_n \in \Delta(X)$ and $m_1, \ldots m_n \in \Delta(X)$:*

$$\sum_i l_i \otimes V_i \preceq \sum_i m_i \otimes V_i \Longleftrightarrow \sum_i u_{\mathcal{V}}(l_i)\mu_{\mathcal{V}}(i) \leq \sum_i u_{\mathcal{V}}(m_i)\mu_{\mathcal{V}}(i).$$

Moreover, both $u_{\mathcal{V}}$ and $\mu_{\mathcal{V}}$ are unique.

Proof. Let $QL_{\mathcal{V}}$ stand for the set of Q-lotteries with base $\mathcal{V}$, and let $\preceq_{\mathcal{V}}$ denote the restriction of preference $\preceq$ to $QL_{\mathcal{V}}$. Each Q-lottery $\sigma = \sum_i l_i \otimes V_i \in QL_{\mathcal{V}}$ can be seen as a 'horse lottery' $f : I \to \Delta(\mathcal{X})$, where $f(i) = l_i$ for any $i \in I$. Moreover, thanks to axioms $\mathbf{A1} - \mathbf{A4}$, the relation $\preceq_{\mathcal{P}}$ satisfies all the Anscombe-Aumann axioms. Therefore, by theorem 13.2 in Fishburn (1970), we obtain a utility function $u_{\mathcal{V}}$ on $\Delta(X)$ and a probability measure $\mu_{\mathcal{V}} \in \Delta(I)$ that achieve the representation stated in Claim 1. The uniqueness of the probability vector $\mu_{\mathcal{V}}$ is also given by this theorem. However, $u_{\mathcal{V}}$ is unique only up to positive affine transformation. But then, we can assume without loss of generality that it is normalized. Normalization gives the uniqueness of $u_{\mathcal{V}}$. ■

Claim 2. *For any Q-lottery σ, there exists an ordinary lottery $l \in \Delta(X)$ such that $\sigma \approx l \otimes H$.*

Proof. Suppose that $\sigma = \sum_i l_i \otimes V_i$. Among the finite set $\{l_i, i \in I\}$ of lotteries there is a best lottery l^* and a worst one l_*, such that $l_* \preceq_L l_i \preceq_L l^*$. Due to the monotonicity axiom **A4**, we have $\sum_i l_* \otimes V_i \preceq \sigma \preceq \sum_i l^* \otimes V_i$. Note that these three Q-lotteries have the same base $\mathcal{V}$. So we can apply Claim 1. By the representation there, we obtain:

$$u_{\mathcal{V}}(l_*) \leq \sum_i u_{\mathcal{V}}(m_i)\mu_{\mathcal{V}}(i) \leq u_{\mathcal{V}}(l^*).$$

By linearity, we obtain $\alpha \in [0,1]$ such that $\sum_i u_{\mathcal{V}}(m_i)\mu_{\mathcal{V}}(i) = \alpha u_{\mathcal{V}}(l_*) + (1-\alpha)u_{\mathcal{V}}(l^*) = u_{\mathcal{V}}(l)$ where $l = \alpha l_* + (1-\alpha)l^*$. Finally, by Claim 1, we obtain $\sigma \approx \sum_i l \otimes V_i$ and, by **A5**, $\sigma \approx l \otimes H$. ■

Claim 3. *$u_{\mathcal{V}}$ is independent of the OD upon which it is built in Claim 2.*

Proof. Due to **A5**, each of the functions $u_{\mathcal{V}}$ represents the preference $\preceq_L$ on the $\Delta(X)$. Therefore (due to the uniqueness part of the von Neumann and Morgenstern theorem) we obtain that all these functions $u_{\mathcal{V}}$ are positive affine transformations of each other. Since they are normalized, they are in fact equal to each other. ■

From now on, we use the notation $\mu_{\mathcal{V}}$ to denote the function mapping each $V_i \in \mathcal{V}$ to $\mu_{\mathcal{V}}(\{i\})$. So, with a slight abuse of notation, we have

Definition. Let us call the number $U_{\mathcal{V}}(\sigma) = \sum_i u(l_i)\mu_{\mathcal{V}}(V_i)$ by *$\mathcal{V}$-utility value* of Q-lottery $\sigma = \sum_i l_i \otimes V_i$ based on OD $\mathcal{V} = (V_i, i \in I)$.

From Claim 1 follows that $\mathcal{V}$-utility values allows to compare Q-lotteries based on $\mathcal{V}$. But we assert that it allows to compare any Q-lotteries.

Claim 4. *Let $\sigma = \sum_i l_i \otimes V_i$ be a Q-lottery with the base $\mathcal{V} = (V_i, i \in I)$ and $\tau = \sum_j m_i \otimes W_j$ be a Q-lottery with the base $\mathcal{W} = (W_j, j \in J)$. Then $\sigma \preceq \tau$ iff $U_{\mathcal{V}}(\sigma) \leq U_{\mathcal{W}}(\tau)$.*

Proof. Due to Claim 2, the lottery σ is equivalent to some lottery $l \otimes H$ and, consequently, to the lottery $\sum_i l \otimes V_i$. Therefore, $U_{\mathcal{V}}(\sigma) = U_{\mathcal{V}}(\sum_i l \otimes V_i) = u(l)$. Similarly, if

τ is equivalent to $m \otimes H$, then its $\mathcal{W}$-utility value is equal to $u(m)$. Now

$$\sigma \preceq \tau \Leftrightarrow l \otimes H \preceq m \otimes H \Leftrightarrow u(l) \leq u(m) \Leftrightarrow U_{\mathcal{V}}(\sigma) \leq U_{\mathcal{W}}(\tau). \blacksquare$$

Claim 5. *For any event V, the number $\mu_{\mathcal{V}}(V)$ does not depend on OD $\mathcal{V}$ (which contains V).* In other words, let $\mathcal{V} = (V_1, \ldots, V_n)$ and $\mathcal{W} = (W_1, \ldots, W_p)$ be two ODs, and assume that $V_1 = W_1 = V$. Then $\mu_{\mathcal{V}}(V) = \mu_{\mathcal{W}}(V)$.

Proof. Let us consider the Q-lottery $l^* \otimes V_1 + \sum_{i=2}^{n} l_* \otimes V_i$ with the base $\mathcal{V}$. Its $\mathcal{V}$-utility is equal to $u(l^*)\mu_{\mathcal{V}}(V_1) + \sum_{i=2}^{n} u(l_*)\mu_{\mathcal{V}}(V_i) = \mu_{\mathcal{V}}(V)$. Now let us define the auxiliary OD $\mathcal{R} = (V, V^{\perp})$ and the corresponding Q-lottery $l^* \otimes V + l_* \otimes V^{\perp}$. Due to axiom **A5** it is equivalent to the first lottery. Therefore its $\mathcal{R}$-utility (which equals $\mu_{\mathcal{R}}(V)$) is, by Claim 4, equal to $\mu_{\mathcal{V}}(V)$. The same applies to $\mathcal{W}$ and gives $\mu_{\mathcal{V}}(V) = \mu_{\mathcal{R}}(V) = \mu_{\mathcal{W}}(V)$. $\blacksquare$

Thus, we have a system of numbers $\mu(V)$, where V runs over $\mathbb{P}(H)$. Obviously, it forms a probability measure on $\mathbb{P}(H)$. And due to Claim 4, the utility $U_{u,\mu}$ represents the preference $\preceq$ on $QL(H)$. The uniqueness of μ follows easily from Claim 1. $\blacksquare$

Theorem 1 establishes that the axioms characterizing expected utility over quantum lotteries are very similar to the ones used by Anscombe-Aumann in a classical uncertainty context. In fact axioms A2 to A4 is a straightforward generalization of their corresponding axioms to the case when lotteries are defined on the same OD. Axiom A5 which links preferences over lotteries defined on different OD is specific to our setting. It is implicitly satisfied in the classical uncertainty context.

5 Reformulation in terms of Hermitian operators

In this section, we start by reformulating all the ingredients of our decision theory in terms of operators. While expressing the theory in terms of events is more intuitive and closer to the standard formulation, operators are easier to work with. Moreover as we address the issue of defining what a quantum lottery conditional on new information means, we cannot but resort to operators. As we show this is without loss of generality in the static context. In the dynamic context, the two formulations are equivalent under axiom A1-A5. Besides operators have a nice dynamic flavor better suited to the non-classical environment. In that context an event "happens" (as the result of a measurement) rather than pre-existed as in the classical model of the world.

5.1 Basic notions in terms of operators

Let V be an event, that is a (closed) vector subspace in H. Let pr_V be the operator of orthogonal projection of H on V. This means that, for any $h \in H$, $pr_V(h)$ is a unique vector in V such that $h - pr_V(h)$ is orthogonal to V. It is easy to check that

a) pr_V is a linear operator;

b) $pr_V(h) = h$ if and only if $h \in V$; so that $pr_V pr_V = pr_V$;

c) $(pr_V(v), w) = (v, pr_V(w))$ for all $v, w \in H$ (this property is called Hermitian).

For simplicity, an operator with the properties a) - c) will be called a *projector*. Conversely, let P be a projector in H. Let us associate to it the event-subspace $V = \{v \in H, Pv = v\}$. We assert that $P = pr_V$. This follows from the fact that $P(w) = 0$ iff w is orthogonal to V which in turn follows from the Hermitian quality of P.

This construction allows to identify events and projectors. In addition

1) Projectors P and Q are orthogonal when $PQ = 0$.

2) The opposite (to P) projector is $E - P$, where E is the identity operator on H.

3) If projectors P and Q are orthogonal to each other then the operator $P + Q$ corresponds to the sum of the events.

4) If projectors P and Q commute then their product PQ is a projector as well and corresponds to the intersection of the events.

In light of the identification between events and operators, we reformulate an orthogonal decomposition (OD) as follows

Definition. An *orthogonal decomposition of the unit* (ODU) is a finite collection $(P_i, i \in I)$ of projectors which satisfies two properties:

(a) $P_i P_j = 0$ for $i \neq j$;

(b) $\sum_i P_i = E$.

Since OD $(V_i, i \in I)$ and ODU $(P_i, i \in I)$ are equivalent objects, we now express an quantum lottery as follows.

A *quantum lottery* $\sum_i x_i \otimes V_i$ with OD $(V_i, i \in I)$ as its *base* and with an accompanying collection of prizes $(x_i, i \in I)$ can be equivalently expressed as $\sum_i x_i \otimes P_i$ where $(P_i, i \in I)$ is an ODU and $P_i = pr_{V_i}$.

5.2 Belief operators

Theorem 1 above shows that the construction of an expected utility may be divided into two independent parts: the utility of roulette lotteries u and the "probabilistic" part expressed by the measure μ. What the utility function concerns everything is clear. In contrast, the probability measure μ on the lattice $\mathbb{P}(H)$ leaves some questions unanswered. Indeed we do not know how to construct such a measure, how many such measures exist and what we can do with them. In this section we provide an answer to these questions. We first recall some basic properties of Hermitian operators.

Definition. A linear mapping (operator) $A : H \to H$ is called *Hermitian* if

$$(Av, w) = (v, Aw)$$

for any $v, w \in H$.

As before, $(Av, v) = (v, Av)$ is a real number for any $v \in H$. The set $\mathbf{Herm}(H)$ of Hermitian operators on H is a vector space, but *over the field* $\mathbb{R}$ *of real numbers.* As we have seen, projectors are Hermitian operators. Therefore any linear combination of projectors with *real* coefficients also is a Hermitian operator. The following theorem, a key result in the theory of Hermitian operators, states that any Hermitian operator is obtained by such a way.

The Spectral Theorem. *Let A be an Hermitian operator on (finite-dimensional) Hilbert space H. Then there exists a (finite) 'spectral' representation*

$$A = \sum_i a_i P_i,$$

where $(P_i, i \in I)$ is an ODU, and a_i are real numbers.

Equivalently, A can be represented by a diagonal matrix (with real diagonal members) in some orthogonal basis of H. An Hermitian operator A is *non-negative* ($A \geq 0$) if $(Av, v) \geq 0$ for all $v \in H$. In terms of the spectral theorem, this is equivalent to all $a_i \geq 0$.

Definition. For a given function $u : \Delta(X) \to \mathbb{R}$, the *shadow operator* of a Q-lottery $\sigma = \sum_i l_i \otimes P_i$ is the Hermitian operator $Op^u(\sigma)$ given by the formula:

$$Op^u(\sigma) = \sum_i u(l_i) P_i.$$

As we will see, the utility of a Q-lottery is fully determined by its shadow operator..

The trace of an operator. For an arbitrary (not necessary Hermitian) operator D on a (finite-dimensional) vector space H, there is the *trace* $\mathbf{Tr}(D)$. More precisely, the trace is defined for a square matrix as the sum of its diagonal members. But it is well known that this definition is correct for operators (that is independently of the choice of basis). This follows from the following remarkable fact: $\mathbf{Tr}(AB) = \mathbf{Tr}(BA)$ for any operators A and B. Of course, the trace of an operator linearly depends on this operator. Finally, the trace of an operator with spectral decomposition $\sum_i a_i P_i$ is equal to $\sum_i a_i rk(P_i)$ which implies the non-negativity of the trace of non-negative operators.

Definition. A *belief operator* is an Hermitian non-negative operator with the trace

1.

Remark. In quantum physics such operators are called "states" or density operators. We call them 'beliefs' (or cognitive state) because they allow constructing subjective probability measures in a most suitable way. Indeed, let D be a belief operator, for an event V, we denote $\mu_D(V) = \mathbf{Tr}(pr_V D)$. It denotes the subjective probability for event V when the cognitive state is D.

Proposition 2.

1) This system of numbers is a probability measure on $\mathbb{P}(H)$.

2) Different belief operators define different probability measures.

3) If $\dim H \geq 3$ then every probability measure on $\mathbb{P}(H)$ has the form μ_D for a (unique) belief operator D.

Proof. 1) Every number $\mu_D(V) = \mathbf{Tr}(PD)$, where $P = pr_V$, is non-negative. Indeed, $\mathbf{Tr}(PD) = \mathbf{Tr}(PPD) = \mathbf{Tr}(PDP)$ but PDP is a Hermitian and non-negative operator. While $\mu_D(H) = 1$ since $\mu_D(H) = \mathbf{Tr}(ED) = \mathbf{Tr}(D) = 1$.

2) Suppose that two belief operators D and D' generate the same measure, $\mu_D = \mu_{D'} = \mu$. Let $A = D - D'$ and let $A = \sum_i a_i P_i$ be a spectral representation of A. Then $\mathbf{Tr}(AD) = \sum_i a_i \mathbf{Tr}(P_i D) = \sum_i a_i \mathbf{Tr}(P_i D') = \mathbf{Tr}(AD')$. Therefore, $\mathbf{Tr}(AD - AD') = \mathbf{Tr}(AA) = 0$. But $\mathbf{Tr}(A^2) = \sum_i a_i^2 rk(P_i)$, from where we have $a_i = 0$ for every $i \in I$ and $A = 0$.

3) It is the famous Gleason's theorem. ■

Fix u and $\mu = \mu_D$. Then let $U = U_{(u,\mu)}$. We have: $U(\sigma) = \mathbf{Tr}(Op^u(\sigma)D)$. So indeed the utility of a Q-lottery is fully determined by its shadow operator.

Collecting all together we obtain[4]

Theorem 2. *Let $\preceq$ be a nice binary preference relation on the set $\mathbf{QL}(H)$. Assume $\dim(\mathcal{H}) \geq 3$. Then there exists an affine utility function u on $\Delta(X)$ and a belief operator*

[4]A similar result can be found in Gyltenberg and Hansen (2012) relying on other axioms. In particular in place of their axioms of Indifference and Separation, we use our No-framing axiom.

D such that $\preceq$ is given by the function $U_{u,D} : \sigma \mapsto \mathbf{Tr}(Op^u(\sigma)D)$. If the preference relation $\preceq$ is not trivial then D is unique and u is unique up to positive affine transformation.

Theorem 2 uses the language of operators to make the representation in Theorem 1 more precise. It allows to fully characterize the probability that captures the decision-maker's subjective beliefs in terms of a belief operator D. The operator $Op^u(\sigma)$ plays the same role as the utility profile in Anscombe-Aumann and the expectation can be defined as the trace of the product $Op^u(\sigma)D$ without loss of generality provided $\dim(\mathcal{H}) \geq 3$.

Example 4. Let $A = Op^u(\sigma)$, the expected value of this lottery when the belief state is D is $\mathbf{Tr}\,(AD)$. For concreteness let us consider the case when D is a pure state that is it corresponds to vector $e \in H$ (of length 1, see Example 2). The belief operator D takes the form of $P_e : x \to (x, e)e$. In this case $\mathbf{Tr}\,(AD) = (Ae, e)$. The quadratic form (Ae, e) is interpreted as the utility value of lottery A when the DM's cognitive state is given by e.

Example 5. Consider again $A = Op^u(\sigma)$ now in the case when $D = E/\dim H$ which represents "complete uncertainty" (cf. example 2). In the case of complete uncertainty the expected utility value of lottery A is equal to $\mathbf{Tr}\,(A)\,/\dim H$, that is the arithmetic average of the eigenvalues of operator A. In the general case the expected utility of A is a convex combination of its eigenvalues. To see this we take the spectral decomposition of $A = \sum_i \lambda_i P_i$ where λ_i are the eigenvalues of A, and P_i the projectors on the eigenspaces. Then $\mathbf{Tr}\,(AD) = \sum_i \lambda_i \mathbf{Tr}\,(P_i D)$, where $\mathbf{Tr}\,(P_i D) \geq 0$ and $\sum_i \mathbf{Tr}\,(P_i D) = 1$.

6 Dynamic consistency: the updating rule

Suppose that the beliefs of a DM are given by belief operator D. That is, our decision-maker believes that the state of the measured quantum system is D. Alternatively, her cognitive state is such that she assigns probabilities to events according to D. In the next section we return to the notion of state which is delicate even in Quantum Mechanics and to the interpretation of D in our context. But for now suppose that she (or someone else) performs some measurement and learns as a result of the measurement that an event W occurred. In short, she receives new information that the system has property W represented by projector P. It is almost evident that her beliefs and preferences on Q-lotteries should change, the question we ask in this section is how should her preferences on quantum lotteries change after receiving that information? The objective is to secure consistency between initial preferences and preferences conditional on new information.

In Quantum Mechanics, it is assumed that the state changes in accordance with the von Neumann-Lüders postulate. More precisely, a system that was in state D transits to the state $D' = PDP/\mathbf{Tr}(PDP)$ as a result of performing a measurement that yields outcome W. The operator PDP is Hermitian and non-negative ($(PDPv, v) = (DPv, Pv) \geq 0$ by force of the nonnegativity of D); its trace is by definition equal to 1. Thus, D' is indeed a state. Here, we need to clarify why $\mathbf{Tr}(PDP)$ is different from zero so we are allowed to divide by this number. As a matter of fact, we understand $\mathbf{Tr}(PDP)$ as the probability to discover event P in cognitive state D. Thus, by analogy with standard Bayesian updating, the von Neumann-Lüders postulate focuses on cases where the state assigns a positive probability to event P. If the trace $\mathbf{Tr}(PDP)$ were equal to 0, that would mean that something happened that has zero probability i.e., an event that is considered impossible under the belief D. That is, the beliefs of our DM captured by the state D are simply incorrect and she has to update in a more fundamental way.

We want to show that in quantum decision theory the beliefs change in the same way. For that, we clearly have to require something. In order to understand what, we return for a minute to the behavior of a classical decision-maker. She has preference between functions (acts) defined on the set S of states of nature; suppose that she learns in addition that the true state lies in some subset $T \subseteq S$. It is quite natural to assume that her new preference depends only on values of these functions on the subset T. That is, only on the restriction of the various functions to T.

We should confess that we do not know how to define such a restriction in the quantum case i.e., to define canonically a lottery $\sigma|W = \sum_j m_i \otimes Q_j$, where (Q_j) form an ODU in the Hilbert subspace W. The natural candidates $Q_i = PP_i$ (where P is the projector on W and PP_i is a Hermetian operator in space W) are generally not projectors and do not commute with each other.

The only case when we can apply this approach without problem is when projector $P = pr_W$ on W commutes with any of the projectors P_i. In this case we can set $\sigma|W = \sum_i l_i \otimes PP_i$ (here we interpret PP_i as an operator on W, because $PP_i(w)$ lies in W). Indeed, PP_i are Hermitian projectors, and their sum is equal to P, the identity operator in W. This definition is consistent with the classical case discussed above. Note finally, that in this case we have

$$Op^u(\sigma|W) = \sum_{i\in I'} PP_i u(l_i) = P(\sum_{i\in I} u(l_i)\ P_i) = POp^u(\sigma).$$

We want to use the property above to define a restriction in general case. Strictly speaking, we will not define the restriction not of a Q-lottery, but of its shadow operator. Since the utility of a lottery only depends on its shadow operator, this is sufficient for our purpose. For any Hermitian operator A on H we define $A|W$ as operator PA, which is an operator in subspace W. As an operator in W, it is Hermitian. Indeed, for any $v, w \in W$,

$$(PAv, w) = (Av, Pw) = (Av, w) = (v, Aw) = (Pv, Aw) = (v, PAw).$$

For some reason, it is easier and more practical for us to work with operators on $\mathcal{H}$. We shall denote with $A||W$ the Hermitian operator PAP in $\mathcal{H}$. This operator has the same restriction to W as A. Moreover, it is equal to 0 on the ortho-complementation $W^\perp$ of W in $\mathcal{H}$.

Definition The *restriction* of Hermitian operator A to a subspace W is defined as the Hermitian operator PAP where $P = pr_W$. We denote it by $A||W$.

Two features justify this definition. First, the restriction is equal to zero on the orthogonal complement of W. Consider a (unitary) vector e orthogonal to W that is $Pe = 0$. Then $(PAPe, e) = (PA0, e) = (0, e) = 0$. Next, the restriction coincides with A on W. Consider again a (unitary) vector e in W, that is $Pe = e$. Then

$$(PAPe, e) = (APe, Pe) = (Ae, e) \,.$$

So the operator $A||W$ coincides with A, when the (pure) states is in W, and is equal to zero when the states are orthogonal to W.

Let us go back to the question concerning the change of in "nice" preference $\preceq$ after learning that event W occurs. The new preference will be denoted as $\preceq_W$. We assume that each of $\preceq$ and $\preceq_W$ is nice and nontrivial. Let (u, D) and (u_W, D_W) be two pairs providing a nice representation as in Theorem 2 for $\preceq$ and $\preceq_W$ respectively.

Our first, "stability" (ST), axiom expresses the hypothesis that the taste for prizes should be stable and independent of information. More precisely:

ST *For any $l, m \in \Delta(X)$, $l \otimes H \succsim_W m \otimes H \iff l \otimes H \succsim m \otimes H$.*

It is easy to see that Axiom ST is equivalent to u and u_W being positive affine transformations of each other. From now on, we will assume without loss of generality that $u = u_W$.

Our second axiom, "consistent updating" (CU) captures the requirement that preferences should remain consistent as the agent acquires information.

CU *Let $\sigma, \tau, \sigma', \tau' \in QL$ such that $Op^u(\sigma') = Op^u(\sigma)\,||W$ and $Op^u(\tau') = Op^u(\tau)\,||W$, then $\sigma \preceq_W \tau$ if and only if $\sigma' \preceq \tau'$.*

First, note that, in this axiom, σ' can be thought of as a Q-lottery that yields the same utilities as σ if W obtains and 0 otherwise. Likewise, τ' can be thought of as a Q-lottery that yields the same utilities as τ if W obtains and 0 otherwise. Therefore, σ' and τ' yield the same utilities over $W^{\perp}$. Now, to understand this axiom, it is helpful to decompose it into two parts. The first part relates to the idea of consistency: since σ' and τ' yield the same utilities over $W^{\perp}$, it is indeed a matter of consistency to require *ex ante* and *ex post* preferences over σ' and τ' to agree ; that is, to impose $\sigma' \preceq_W \tau' \Longleftrightarrow \sigma' \preceq \tau'$. Roughly, this first part corresponds to what Ghirardato (2002) calls *dynamic consistency* (see his Axiom 2). Moreover, note that by construction σ and σ' yield the same utilities over W, therefore it is natural to require indifference conditional on W. Since the same reasoning applies to τ and τ', we require $\sigma' \sim_W \sigma$ and $\tau' \sim_W \tau$. This second argument corresponds to *consequentialism* (see Ghirardato's Axiom 7). Combining these two arguments, we finally get the equivalence between $\sigma \preceq_W \tau$ and $\sigma' \preceq \tau'$, as required by CU. In example 6 below we return to this discussion more formally.

Theorem 3. *Assume that* $\mathbf{Tr}(PDP) > 0$*. Then Axiom CU holds if and only if the belief operator is updated into* $D_W = (D||W)\,/\mathbf{Tr}(D||W) = PDP/Tr(PDP)$.

Proof. Assume Axiom CU. Then, for any σ,$\tau \in QL$, denote $Op^u(\sigma) = A$ and $Op^u(\tau) = B$, for $A, B \in \mathbf{Herm}(H)$. Let $\sigma', \tau' \in QL$ *such that* $Op^u(\sigma') = A||W$ *and* $Op^u(\tau') = B||W$:

$$\sigma \preceq_W \tau \quad \Longleftrightarrow \quad \sigma' \preceq \tau' \quad \Longleftrightarrow \quad \mathbf{Tr}((A||W)D) \leq \mathbf{Tr}((A||W)D)$$

But note that $\mathbf{Tr}(PAPD) = \mathbf{Tr}(A(PDP))$, and similarly for B. Moreover, $\mathbf{Tr}(PDP) > 0$. Therefore, and using linearity as well, we have

$$\sigma \preceq_W \tau \quad \Longleftrightarrow \quad \mathbf{Tr}(A(PDP)/\mathbf{Tr}(PDP)) \leq \mathbf{Tr}(B(PDP)/\mathbf{Tr}(PDP)).$$

By the uniqueness part of the Theorem 2, we obtain $D_W = (PDP)/\mathbf{Tr}(PDP)$. Now assume $D_W = (PDP)/\mathbf{Tr}(PDP)$. Note that, for any $A \in \mathbf{Herm}(H)$, we have

$$\mathbf{Tr}(AD_W) = \mathbf{Tr}((A||W)D)/\mathbf{Tr}(PDP).$$

This follows from the equality $\mathbf{Tr}(APDP) = \mathbf{Tr}(PAPD)$, which is true due to the commutativity of the trace. Axiom CU is then a simple consequence.■

It is well-known (Ghirardato 2002) that Bayesian updating of beliefs secures dynamic consistency of preferences in a classical uncertainty context. Theorem 3 establishes that the von Neumann-Lüders postulate similarly secures dynamic consistency of preferences (axiom CU) in a non-classical (quantum) uncertainty environment.

In the remaining of the paper we use the "shadow" operator expression for lotteries: $A = Op^u(\sigma)$ and as defined in Theorem 2 the utility of such a lottery under belief D is equal to $U(A) = \mathbf{Tr}(AD)$. We next comment on our result in Theorem 3.

Generally, new information arises as the outcome of some "informational" measurement. Some measurement, represented by an ODU $(P_i, i \in I)$ is performed. If, as the result of this measurement, we obtain outcome i, the system transits into subspace $W = Im(P_i)$ and, in accordance with Theorem 3, the belief-state D changes (is updated) into $D_i = P_iDP_i/\mathbf{Tr}(P_iD)$. Note that the number $\mathbf{Tr}(P_iD)$ is precisely the probability (under belief D) for the realization of outcome i when performing our measurement; we denote it by p_i. Without loss of generality we can assume that these numbers are strictly positive, except for "impossible" results.

The utility of "lottery" A, after the DM received the information about the realization of outcome i, is now equal to $U_i(A) = \mathbf{Tr}(AD_i)$. It may clearly be either larger or smaller than the initial $U(A)$. And as we shall see the connection between these two utilities i.e., *ex-ante* and *ex-post* is not straightforward in general except for two cases: when our measurement $(P_i, i \in I)$ is compatible with either the lottery or the belief state i.e., when all the $P_i, i \in I$ commute with either operator A or D.

Proposition 3. *Assume that our measurement is compatible with either operator A or D. Then we have the following formula*

$$U(A) = \sum_i \mathbf{Tr}(P_iD)\,\mathbf{Tr}(AD_i) = \sum_i p_iU_i(A).$$

Proof. Assume that A commutes with P_i, $P_iA = AP_i$. Since $E = \sum P_i$ we have

$$U(A) = \mathbf{Tr}(AD) = \mathbf{Tr}(\sum_i P_iAD)$$

Since the P_i are projectors we have that $P_iAD = P_iP_iAD = P_iAP_iD$. Therefore $\mathbf{Tr}(\sum_i P_iAD) =_i \mathbf{Tr}(P_iAP_iD) =_i \mathbf{Tr}(AP_iDP_i) = p_i\mathbf{Tr}(AD_i)$. This gives us the formula in Proposition 3. Assume now that D commutes with the P_i. Then we have

$$\begin{aligned} U(A) &= \mathbf{Tr}(AD) = \mathbf{Tr}(A(\sum_i P_iD)) =_i \mathbf{Tr}(AP_iP_iD) = \\ {}_i\mathbf{Tr}(P_iAP_iD) &= {}_ip_i\mathbf{Tr}(AD_i) =_i p_iU(A_i) .\blacksquare \end{aligned}$$

To put it differently the *ex-ante* utility is equal to the weighted sum of *ex-post* utilities with the weights equal to the probabilities p_i for the different outcome i when performing our (intermediary) measurement. From this formula we see immediately that we obtain a relationship reminding of the "Sure Thing Principle" referred to as "recursive dynamic consistency" in its dynamic version. The idea is that if lottery A provides *ex-post* a higher utility than B whatever the outcome i of an intermediary measurement, then lottery A is preferred to B *ex-ante*. We want emphasize that this is true only under the condition that all the $P_i, i \in I$ commute with either operator A or D. In the general case "recursive dynamic consistency" is violated in our setting as we show next with a simple two-dimensional example.

Example 6. Let H be a two dimensional Hilbert space with orthonormal basis (e_1, e_2). Let A be a projector on e_1, i.e., an operator of the form $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Consider A as a lottery that gives a utility equal to 1 in state e_1 and 0 otherwise. Consider another lottery-operator $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ that gives utility 1 in state e_2 and 0 otherwise. Assume now that our DM's belief-state is given by $D = A$. Then clearly the expected utility of lottery A is equal to 1 and the expected utility of B is equal to zero. So our DM strictly prefers A to B.

Assume now that we perform a measurement defined by the ODU (P_1, P_2) where $P_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$. If the outcome of the measurement is 1, the updated belief-state is given by operator $D_1 = P_1 D P_1 / \mathbf{Tr}\left(P_1 D P_1\right)$,which as can be seen easily is equal to P_1. The expected utility in the belief-state is $U_1\left(A\right) = \mathbf{Tr}(AD_1) = \mathbf{Tr}(AP_1) = 1/2$. And similarly if we obtain the complementary result 2, the belief-state is updated to $D_2 = \frac{P_2 D P_2}{\mathbf{Tr}(P_2 D P_2)}$ and the corresponding expected utility is $U_2\left(A\right) = \mathbf{Tr}(AD_2) = \mathbf{Tr}(AP_2) = 1/2$. So we see that for any one of the two possible outcomes the value of the A lottery goes from 1 to 1/2. With the same reasoning we obtain that $U_1\left(B\right) = \mathbf{Tr}(BD_1) = \mathbf{Tr}(BP_1) = 1/2 = U_2\left(B\right)$. So the two lotteries A and B yield the same expected utility. This violates "recursive dynamic consistency": lottery A is *ex-post* indifferent to be B whatever the outcome of the measurement, yet *ex-ante* it is strictly preferred. This violation occurs because the intermediary measurement (P_1, P_2) is incompatible with either A or B and D.

This simple example allows illustrating yet one feature that lacks counter-part in the classical model. Imagine that we perform the measurement described above but our DM is not informed of the result. She only knows the measurement has been made. In the classical world such an information does not affect the DM's belief or the expected value of the lotteries. However in our "quantum" context the DM understands that for any of the two outcome (1 or 2) the expected value of lottery A has changed from 1 to 1/2. Therefore, independently of her (lack of) knowledge about the outcome of the measurement, she will revise her belief-state only because she knows that this specific measurement has been performed. The new belief-state is $D' = p_1 D_1 + p_2 D_2 = 1/2E$ which corresponds to "uniform ignorance". And in this belief-state the expected utility of lottery A is equal to $\mathbf{Tr}\left(AD'\right) = 1/2\mathbf{Tr}\left(AE\right) = 1/2$.

7 Discussion: quantum cognition and rationality

In this section we return to the question as to when extending expected utility theory to a non-classical uncertainty environment may be valuable i.e., what kind of situations does it address.

We start with a short discussion of the notion of belief-state or cognitive state. It must be emphasized that even in Quantum Mechanics, the notion of state is far from transparent. Does the state pertain to the system? Or does it pertain to the observer and encapsulates his knowledge about a system? In the classical world everything is clear. The system is endowed with properties and our beliefs about the system reflect exclusively our (incomplete) information about that system. A key insight of QM is that we cannot as in the classical world always separate the system from its observer and the measurement operations he undertakes. As in the classical world, the cognitive state encapsulates our information about the system but unlike the classical world our attempts to learn more by means of measurements affect the system so its properties change. This applies when the system itself is quantum e.g., some subatomic particle. But why is that relevant to decision theory? Indeed lotteries are generally not defined on states of subatomic particles.

Quantum cognition

As mentioned in the Introduction, there has been a number of works in decision theory both theoretical and experimental developing the idea that the (quantum) indeterminacy of *preferences* can explain a number of behavioral anomalies. Although this is different from what we do in this paper it is relevant to our current concern in the following way. The uncertainty relevant to our decision-maker may concern other agents' preferences e.g., in an interactive i.e., game situation. The type indeterminacy approach (Lambert-Mogiliansky, Zamir, and Zwirn (2009)) identifies the choices made by other agents as the outcomes of some measurement of their preferences. That is the choice made by another player is the new information upon which our decision-maker must update his belief-state. The present work characterizes how a player should update her beliefs about the other agents' indeterminate preferences. In so doing the present paper is a contribution to an emerging literature on games with quantum-like players.

A most important field of application lies within quantum cognition where agents' quantum-like characteristics apply to their beliefs or representation/perception of the world. Interestingly it is also consistent with an interpretation of quantum mechanics according to which the weird features of quantum mechanics do not belong to "the world out there" but to the world that we can access with our mind and instruments. In fact it is argued that whether or not there exists an objective world is not relevant because the only ways to access it induces a relationship between us and the world where the two are impossible to disentangle (cf. Bitbol (2009)). Most importantly for us is that the world that matters to decision-making is the perceived world, that is the world that we can access with our human mind and measurement instruments.

It is uncontroversial to assert that in order to assess the world we build a representation of it, a "represented world" which is a mental construct. In classical standard theory, the represented world reflects our incomplete knowledge about the world expressed in our beliefs and these beliefs (should) evolve according to Bayes' rule in response to new information. In contrast the quantum cognitive approach is based on two observations: 1. People have difficulties to represent themselves a complex object. What people do is to consider a complex object from different perspective - one at a time; 2. People have difficulty in combining perspectives i.e., to synthesize all relevant information into one single coherent representation of the complex object. Quantum cognition models these difficulties by analogy with incompatible properties in Quantum Mechanics: different perspectives may be incompatible in the mind but they complement each other by contributing with information. Non-commutativity in information processing (updating) results from the conjunction of on the one hand the incompatibility of perspectives and on the other hand the oneness of the mental picture: when you learn something in one perspective, this affects your whole representation so you actually may lose information regarding aspects previously "known" in other incompatible perspectives: learning a new feature perturbs the whole mental construct. One example is in the article about doctors' diagnosis mentioned in the Introduction.[5] Let us see how it connects to the subject matter of this article.

The experiment started with the doctors learning the patient chief complaint. This created an initial and common cognitive state D and associated common preferences. Presumably they all wanted to see the patient recover health. Thereafter one group learned the results (W) of a physical examination ("hot" data) and the other group the result (V) of laboratory test ("cold" data). In our terminology, this amounts to observing the outcomes of the measurement of two possibly incompatible mental perspectives. The new information modified the doctors' initial cognitive state into D_1' for the first and D_2' for the second group. The two groups were then given the information received by the other group and again revised their belief-state into $D_1^{"}$ and $D_2^{"}$ respectively. Thereafter they had to decide about the likelihood of urinary track infection implying a recommendation for treatment i.e., a decision with random payoff (a Q-lottery). The experiment shows that the two groups decided significantly differently. The doctors ended up revealing different preferences solely due to the order in which they processed information. In our setting this can happen when V and W belong to incompatible perspectives represented by non-commuting operators. As a consequence of updating according Theorem 3 we can have $D_1^{"} \neq D_2^{"}$ implying distinct associated revealed preferences for each group. The results in this experiment are thus consistent with the incompatibility in the mind of the "cold" laboratory data perspective and the "hot" physical examination perspective.[6]

The concept of rationality used in economics involves an implicit assumption of the existence of an independent reality. Quantum Mechanics taught us that whether or not

[5] Similar effects have been exhibited in experiment related to how judges' and jury's decisions. The decisions were shown to be significantly dependent on the order in which argument were presented e.g., first defense and than accusation or the other way around.

[6] Other experiments have shown that even a person's physiological response to e.g., a pain stimuli can depend on similar (contextual) factors (see e.g., Zimbardo et al. 1969).

such reality exists may not be possible to assess. This is because we cannot access reality independently from our efforts to measure it and those efforts can perturb it: information is contextual that is relative to the measurement performed on the system. An event that is true in the sense that it has been revealed as the result of some measurement may very well cease to be true in another context that is after another measurement. In our setting this translates into the failure of recursive dynamic consistency as shown in section 6. It may seem odd that our consistency axiom CU allows for such departures. But the appeal of recursive dynamic consistency is based upon the implicit assumption that "the act the agent performs has no effect on the resolution of uncertainty" (cf. Fishburn in the Introduction). However, the resolution of uncertainty is - in our setting - affected by the act that is selected and the measurements it entails (as well as by other measurements) performed to acquire new information. Once this is taken into account, recursive dynamic consistency loses much of its appeal. Our results demonstrate that an individual can behave rationally that is dynamically consistently in a world that she represents himself as non-separable from "the act she selects" (cf Fishburn 1970). In future research, we aim at exploring further the implications for human decision-making of a full-fledged concept of rationality based on non-classical(quantum) logic.

8 Concluding remarks

In this paper we have established rules for rational choice behavior in a non-classical uncertainty environment. We introduced the concept of quantum lottery and formulated sufficient and necessary conditions for choice behavior to be representable by an expected utility function. We also derived from a requirement of dynamic consistency an updating rule that secures that choice behavior conditional on new information is consistent with ex-ante contingent preferences.

We found that most of the classical axioms of decision theory carry over to the context of quantum lotteries. This is because all but one axiom can be formulated in terms of a single orthogonal decomposition of the state space. When considering a single orthogonal decomposition, quantum lotteries operating in the Hilbert space are equivalent to roulette lotteries in a classical state space. This equivalence mirrors the equivalence between classical measurements and compatible quantum measurements. An additional axiom is required to secure that the probability for any specific event does not depend on the particular lottery that it belongs to. The necessity to impose that axiom stems from the fact that while it is trivially true in the classical world, it is not necessary so in our more general setting.

A most interesting result is that the von Neumann-Lüders postulate which is central to Quantum Mechanics and informs about the impact of a measurement on the state of a system can be derived from a consistency requirement on choice behavior. When the belief-state (cognitive state) is updated according to the postulate, the agent conditional preferences reflect a single preference order. In order to establish that result we had to confine ourselves to a restricted class of preferences i.e., those satisfying our axioms. This restriction is needed because the concept of conditional lottery is not well-defined for general quantum lotteries. It is however well-defined for Hermitian operators which represent quantum lotteries satisfying our axioms. Interestingly, we find that in contrast with classical subjective expected utility theory, the dynamic consistency of preferences does not entail the so-called recursive dynmamic consistency. This distinction is an expression of the fundamental distinction between the two settings namely that the resolution of uncertainty depends on the operation(s) performed to resolve it.

This exercise contributes to decision theory by extending expected utility theory to non-classical (quantum) uncertainty. It also contributes to behavioral economics by providing further foundations for the quantum cognitive approach. As we argue in the Introduction and in the Discussion this approach is facing a growing recognition due to its success in explaining behavioral anomalies as well as its deeper meaning and psychological interpretation.

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